Math 358 – Foundations of Mathematics **Fall 2008**

9-18-08 Class Notes

Sections 2.1, 2.2
2.1: 10,
2.2: 22, 24, 31, 32

Set Theory

Two of the most fundamental concepts common to virtually all braches of mathematics are sets and functions. In this set of notes, we'll discuss the former.

Definition: A ***set*** is any collection of objects. If S is a set and x is an object that belongs to S, then we say x is an ***element*** (or ***member***) of S. This is denoted by $x \in S$. If A and B are sets and every element of A is also an element of B, then we say that A is a ***subset*** of B (denoted $A \subseteq B$). If $A \subseteq B$ and $B \subseteq A$, then the sets A and B are ***equal*** (obviously denoted by $A = B$). Finally, if $A \subseteq B$ but $A \neq B$, then we say that A is a ***proper subset*** of B (denoted by $A \subset B$).

Example 1: Let $S = \{x : x \text{ is a prime number}\}$. Then $7 \in S$ and $10 \notin S$.

Definition: The set with no elements is called the ***empty set*** and is denoted by $\varnothing$. If a set A has a finite number of elements, the number of elements in A is called the ***cardinality*** of A and is denoted by $|A|$.

Example 2: Let $A = \{x : x \text{ is a vowel}\}$, $B = \{x : x \text{ is a positive divisor of } 10\}$, and $C = \{x : x \text{ is a baseball player who struck out against Nolan Ryan}\}$. Then $|A| = 5$, $|B| = 4$, and $|C| = 1176$.

Note in the previous definition that we said "THE set with no elements is called the empty set." This implies that there is only one such set. This is established by the following lemma.

Lemma: If $A = \varnothing$ and $B = \varnothing$, then $A = B$. In other words, there is only one empty set.

[Our text makes the point that this lemma actually only guarantees that we have AT MOST one empty set. But maybe there are no empty sets. They then state the axiom that there is an empty set. I'm taking that fact as obviously true – consider the set of baseball players who struck out against ME.]

Note that every nonempty set has at least two subsets: itself and the empty set. In fact, we can say even more than this. But first we need another definition.

Definition: Let S be a set. The set of all subsets of S is called the ***power set*** of S and is denoted by $\mathscr{P}(A)$.

Theorem 1: If $|S| = n$, then $|\mathscr{P}(S)| = 2^n$.

Example 3:

(a) If $A = \{1,2\}$, then $\mathscr{P}(A) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.

(b) If $B = \{a,b,c\}$, then $\mathscr{P}(B) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

(c) If $S = \varnothing$, then $\mathscr{P}(S) = \{\varnothing\}$.

In most problems, we only wish to consider elements of a certain type. Perhaps we are only dealing with odd integers, positive real numbers, or students at Nicholls State University. This is called the ***universe of discourse*** and usually denoted by the set U. For that problem, all sets are assumed to be subsets of that universal set.

Definition: Let A and B be sets. The ***union*** of A and B is the set of all objects that belong to A or B (denoted by $A \cup B$). The ***intersection*** of A and B is the set of all objects that belong to both A and B (denoted by $A \cap B$). The ***difference*** of A and B is the set of all objects that belong to A but do not belong to B (denoted by $A - B$). Finally, the ***complement*** of A is the set of all objects not in A (denoted by A').

Example 4: Let $U = \{1,2,3,\ldots 10\}$, $A = \{2,4,8,9\}$, and $B = \{1,2,3,4\}$. Then $A \cup B = \{1,2,3,4,8,9\}$, $A \cap B = \{2,4\}$, $B - A = \{1,3\}$, $B' = \{5,6,7,8,9,10\}$

Here are a slew of "obvious" facts. Let A, B, and C be sets.

(1) $A \cup \varnothing = A$
(2) $A \cap \varnothing = \varnothing$
(3) $A \cup U = U$
(4) $A \cap U = A$
(5) $A' = U - A$

(6) $A \subseteq A \cup B$ and $A \cap B \subseteq A$
(7) $A \cup B = B \cup A$ and $A \cap B = B \cap A$
(8) $(A')' = A$